JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **207**, 475–495 (1997)
ARTICLE NO. AY975296

On the Vlasov–Poisson–Fokker–Planck Equations with Measures in Morrey Spaces as Initial Data

José A. Carrillo and Juan Soler*

Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada,

Received March 26, 1996

In this paper the existence of weak solutions, local and global in time solutions for small initial distribution of particles, for the three-dimensional Vlasov–Poisson–Fokker–Planck system with measures as initial data is obtained. Also, the uniqueness and stability for these solutions is analysed. The distribution of particles to be considered is the measures with special decay contained in Morrey spaces.

1. INTRODUCTION

The goal of this paper is to extend the existence theory for the three-dimensional Vlasov–Poisson–Fokker–Planck (VPFP) equations to data which are measures and to obtain *global in time* estimates of the solutions. Since Morrey spaces for measures are appropriate for studying some interesting singular structure of the distribution of particles, such as the particle sheet, our interest in this paper is centred on the analysis of the Cauchy problem associated with the VPFP system with initial data in such spaces of measures. To our knowledge there are no results of existence in the previous literature for the three-dimensional VPFP equations with initial data as measure. In the framework of this problem a first result was recently shown for the one-dimensional case by A. Majda and Y. Zheng [14].

Since the VPFP system is not symmetric with respect to the variables (velocity and position) we must obtain some regularity properties of the

*E-mail address: jsoler@ugr.es.

0022-247X/97 $25.00

mass density. The reason for this is that to conclude that the potential is in L^∞ we need to know certain properties of the mass density which, in principle, is only known to be a positive Radon measure. Once we know the potential is in L^∞ the problem can be formulated in meaningful terms. In the literature this question on regularity of the potential is usually treated using the conservation of some moments for the distribution of particles. In this paper we solve this problem assuming that the initial mass density is contained in a suitable Morrey space of measures. The choice of this space assures the L^∞-regularity of the potential. However, with this method we can obtain only the existence of solutions locally in time or a global existence when the data are small enough. Nevertheless, we deduce *global in time estimates* and an estimate on the decay of E which are not available previously even in the L^p case. In fact, these estimates have been used to study the asymptotic behavior of global in time solutions of the VPFP system (see [4, 5]) whose existence is established in this paper at least for small initial data.

The nonlinear VPFP system in three dimensions can be written, on $(0, T) \times \mathbb{R}^6$, as

$$L[f] + \mathrm{div}_v(Ef) \stackrel{\mathrm{def}}{=} \frac{\partial f}{\partial t} + (v \cdot \nabla_x) f + \mathrm{div}_v((E - \beta v) f) - \sigma \Delta_v f = 0, \tag{1.1}$$

where the force field E is given by

$$E = K * \rho(f), \tag{1.2}$$

and K is the gradient of the fundamental solution of the Laplacian in dimension three. Here, $f(t, x, v) \geq 0$ is the scalar distribution of particles in a plasma with respect to the position and velocity, $E(t, x)$ is the electrostatic or gravitational vector force, and $\rho(f)$ is the mass density of f, i.e.,

$$\rho(f)(t, x) = \theta \int_{\mathbb{R}^3} f(t, x, v)\, dv,$$

with $\theta = \pm 1$ and where $\beta \geq 0$ and $\sigma > 0$ are constants which are related to the collision between particles. We refer to [13] for the physical meaning of this model in the case of stellar dynamics ($\theta = -1$) or plasma physics ($\theta = 1$). We consider the initial value VPFP problem for Eqs. (1.1) and (1.2) with initial data f_0.

The VPFP system has been extensively studied in the last years: classical solutions in [10, 13], L^p weak solutions in [3, 12], and the regularity for weak solutions in [1, 2]. The existence of global measure solutions to VPFP in one space dimension is analyzed in [9, 14]. In an interesting paper [14], the relationship with the two-dimensional Euler equation with vortex sheet initial data is crucial in order to study the one-dimensional Vlasov–Poisson equation. In [9] some relevant explicit solutions, which show the phenomena of singularity formation in finite time, are constructed.

The consideration of measures as initial data is interesting in order to include initial data which are relevant from a physical point of view, for example, concentrations of the density in the plasma on a line or on a surface (see [6, 7, 11] for related phenomena in Fluid Mechanics). The mathematical formulation of these concentration phenomena can be included in suitable Morrey spaces.

Let us recall the definition of Morrey spaces. Let $B(a, r)$ be the euclidean ball of center a and radius r. A Radon measure μ lies in the Morrey space $\tilde{L}_p(\mathbb{R}^N)$ if μ satisfies

$$TV(\mu)(B(a,r)) \leq Cr^{N/p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1$$

independently of a and r. $\tilde{L}_p(\mathbb{R}^N)$ is a Banach space endowed with the norm

$$\|\mu\|_{\tilde{L}_p(\mathbb{R}^N)} = \sup\left\{r^{-N/p'} TV(\mu)(B(a,r)) \colon a \in \mathbb{R}^N, r > 0\right\},$$

where $1 \leq p \leq \infty$ and $TV(\mu) = |\mu|$ is the total variation of the measure μ. Note that $\tilde{L}_1(\mathbb{R}^N)$ is the space of Radon measures of finite total variation and $\tilde{L}_\infty(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$.

Let us explain the kind of initial data that we will consider in this paper. Let $f_0 \in \tilde{L}_1(\mathbb{R}^6)$. We define the initial density ρ_0 by $\rho_0(A) = f_0(A \times \mathbb{R}^3_v)$ for all bounded Borel sets A. It is clear that $\rho_0 \in \tilde{L}_1(\mathbb{R}^3)$.

To prove the existence of a solution to our problem, as it is analyzed later, it will not be sufficient to assume $\rho_0 \in \tilde{L}_p(\mathbb{R}^3)$ because the linear part of (1.1) is not symmetric and is a second order degenerate parabolic operator with respect to the velocity variables. These problems lead to estimate functions of the form

$$\rho_h(f_0)(x) = \int_{\mathbb{R}^3} f_0(x - hv, v)\, dv, \qquad \text{with } h \in \mathbb{R}. \tag{1.3}$$

Thus, a uniform control of the $\tilde{L}_p(\mathbb{R}^3)$ norm of the functions $\rho_h(f_0)(x)$, for $h \in \mathbb{R}$, will be necessary. Let us introduce a modified Morrey space which includes the functions such that $\rho_h(f_0(x) \in \tilde{L}_p(\mathbb{R}^3)$, for all $h \in \mathbb{R}^3$. Let $B(a, r, h)$ be the set of points $(x, v) \in \mathbb{R}^6$ satisfying $\|x + hv - a\| \leq r$. Assuming that f_0 belongs to $\tilde{L}_1(\mathbb{R}^6)$ we can write the $\tilde{L}_p(\mathbb{R}^3)$ norm of

$\rho_h(f_0)$ by means of a change of variables to obtain

$$\int_{B(a,r)} \int_{\mathbb{R}^3} f_0(x - hv, v)\, dv\, dx$$
$$= \int_{B(a,r,h)} f_0(x, v)\, d(x, v) = TV(f_0)(B(a, r, h)).$$

Let $M\tilde{L}_p(\mathbb{R}^6)$ be the space of the Radon measures μ such that the quantity

$$\|\mu\|_{M\tilde{L}_p(\mathbb{R}^6)} = \sup\{r^{-3/p'} TV(\mu)(B(a, r, h)) : h \in \mathbb{R}^+, a \in \mathbb{R}^3, r > 0\} \tag{1.4}$$

is finite, where $1 \leq p \leq \infty$. We will denote $ML_p(\mathbb{R}^6) = M\tilde{L}_p(\mathbb{R}^6) \cap L^1_{\mathrm{loc}}(\mathbb{R}^6)$. The modified Morrey spaces, $M\tilde{L}_p(\mathbb{R}^6)$, are a variation of the classical Morrey spaces. Note that $ML_\infty(\mathbb{R}^6)$ is the space of locally integrable functions f_0 whose densities $\rho_h(f_0)$ are bounded independently of h.

It can be proved in a standard way (see [7]) that $M\tilde{L}_p(\mathbb{R}^6)$, $ML_p(\mathbb{R}^6)$ are Banach spaces endowed with the norm (1.4). Moreover, if $\mu \in M\tilde{L}_p(\mathbb{R}^6) \cap M\tilde{L}_q(\mathbb{R}^6)$, then $\mu \in M\tilde{L}_r(\mathbb{R}^6)$, with r satisfying $1/r = (1-\theta)/p + \theta/q$, and

$$\|\mu\|_{M\tilde{L}_r(\mathbb{R}^6)} \leq \|\mu\|^{1-\theta}_{M\tilde{L}_p(\mathbb{R}^6)} \|\mu\|^{\theta}_{M\tilde{L}_q(\mathbb{R}^6)}. \tag{1.5}$$

From now on, we will assume that $f_0 \in \tilde{L}_1(\mathbb{R}^6) \cap M\tilde{L}_{p_0}(\mathbb{R}^6)$, for some p_0 to be specified. Set $\pi(f_0)$ as maximum of the norms of f_0 on these spaces. Let us remark that the classical hypotheses of the boundedness of some velocity moments of the initial data (see [1, 12]) are replaced here by the assumption $f_0 \in M\tilde{L}_p(\mathbb{R}^6)$. In the last section we will connect both hypotheses.

We will denote by $C_w([0, T], \tilde{L}_1(\mathbb{R}^6))$ the space of the continuous functions defined from $[0, T]$ on $\tilde{L}_1(\mathbb{R}^6)$ endowed with the weak-★ topology of bounded measures. We will aim for weak solutions $f \in C_w([0, T], \tilde{L}_1(\mathbb{R}^6))$ of the VPFP problem, i.e., solutions in the sense of distributions on $(0, T) \times \mathbb{R}^{2N}$ of Eq. (1.1) with E given by (1.2) and $f(0, \cdot) = f_0$. This definition is meaningful when we have that Ef is locally integrable on $(0, T) \times \mathbb{R}^{2N}$.

The main results of this paper is the following one.

THEOREM 1.1. *Given a positive measure f_0 as initial data verifying $f_0 \in \tilde{L}_1(\mathbb{R}^6) \cap M\tilde{L}_{p_0}(\mathbb{R}^6)$, with $p_0 \geq 9/4$, then the VPFP problem has a weak* *solution* $f \geq 0$ *in a time interval* $[0, T]$ *with* $T < T^*$ *satisfying*:

(i) $f \in L^\infty(0, T; \tilde{L}_1(\mathbb{R}^6) \cap M\tilde{L}_{p_0}(\mathbb{R}^6))$. *Moreover*, $f(t, \cdot)$ *is an integrable function for any* $t > 0$ *and* $f \in BC((0,T], ML_r(\mathbb{R}^6))$, *for some* $r > 3$ *close enough to* 3.

(ii) *If* $p_0 \geq 3$, *then* $E \in L^\infty(0, T; L^\infty(\mathbb{R}^3)^3)$ *and* $E \in BC((0, T], L^\infty(\mathbb{R}^3)^3)$. *If* $p_0 < 3$, *then* $t^{(9/2p_0)-(3/2)} E \in BC((0, t], L^\infty(\mathbb{R}^3)^3)$.

(iii) $f(t)$ *goes to* f_0 *when t goes to* 0 *weak-*★ *in the topology of bounded measures, in fact f belongs to* $C_w([0, T], \tilde{L}_1(\mathbb{R}^6))$.

(iv) *The solution f verifies, for any* $t > 0$,

$$\int_{\mathbb{R}^3} f(t, x, v)\, dx\, dv = \|f_0\|_{\tilde{L}_1(\mathbb{R}^6)}.$$

Here, T^* *depends on* $\pi(f_0)$ *and on the constants* σ, β. $T^* = \infty$ *for small initial data with* $t^{1/2}E \in L^\infty(0, \infty; L^\infty(\mathbb{R}^3)^3)$ *and* $f \in L^\infty(0, \infty; M\tilde{L}_{9/4}(\mathbb{R}^6))$.

In the next section, we will study some properties of the fundamental solution of the linear equation $L[f] = 0$ and the relation of this fundamental solution with the modified Morrey spaces $M\tilde{L}_p(\mathbb{R}^6)$. In Section 3, we will analyze the main properties of the solutions of an approximate sequence of problems and we obtain some important a priori estimates on this sequence of approximate solutions. In Section 4, we will prove Theorem 1.1 and we derive results of uniqueness, stability, and decreasing at infinity of the force field. And finally, in Section 5, we will establish the relation of the Morrey-type hypotheses with the classical hypotheses of the boundedness of moments.

2. FUNDAMENTAL SOLUTION AND THE MODIFIED MORREY SPACES

In this section we study the linear operator L in the N-dimensional case. Using the results of [1, 13] the fundamental solution of the operator L is written as

$$G(t, x, v, \xi, \nu) = G_0\left(t, x - \xi - \left(\frac{1 - e^{-\beta t}}{\beta}\right)\nu, v - e^{-\beta t}\nu\right), \quad (2.1)$$

where $x, v, \xi, \nu \in \mathbb{R}^N$, $t \geq 0$, and

$$G_0(t, x, v) = \frac{1}{(4\pi\sigma)^N D(t)^{N/2}} e^{-(1/4\sigma)\varphi_0(t,x,v)},$$

$$D(t) = \frac{1}{\beta^2}\left[\frac{\eta(2t)}{2} t - \eta(t)^2\right] \quad \text{and} \quad \eta(t) = \frac{1 - e^{-\beta t}}{\beta}.$$

An explicit formula for φ_0 is developed in [13] with

$$\varphi_0(t, x, v) = \frac{1}{D(t)} \int_0^t |\eta(s)v - e^{-\beta s} x|^2 \, ds.$$

For any $\mu \in \tilde{L}_1(\mathbb{R}^{2N})$ we will denote by $G[\mu]$ the operator

$$G[\mu](t, x, v) = \int_{\mathbb{R}^{2N}} G(t, x, v, \xi, \nu) \, d|\mu|(\xi, \nu). \tag{2.2}$$

The next lemma contains the properties of G that will be useful in the next section. These properties have been proved in [13].

LEMMA 2.1. *The fundamental solution G of* (1.1) *verifies*

(*i*)
$$\int_{\mathbb{R}^{2N}} G(t, x, v, \xi, \nu) \, d(\xi, \nu) = e^{N\beta t}, \; \int_{\mathbb{R}^{2N}} G(t, x, v, \xi, \nu) \, d(x, v) = 1.$$

(ii) *For any $t \geq 0$ and $x, v, \xi, \nu \in \mathbb{R}^N$, we obtain the estimate*

$$|\nabla_\nu G(t, x, v, \xi, \nu)| \leq C(\sigma t)^{-1/2} G\left(t, \frac{x}{2}, \frac{v}{2}, \frac{\xi}{2}, \frac{\nu}{2}\right).$$

(iii) *Setting*

$$d_\lambda(t) = \int_0^t \eta_\lambda(\tau)^2 \, d\tau, \qquad \eta_\lambda(t) = \frac{1 - e^{-\beta t}}{\beta} + \lambda e^{-\beta t},$$

$$\mathcal{N}(x) = \frac{1}{(2\pi)^{N/2}} e^{-|x|^2/2},$$

for any $\lambda \geq 0$, we obtain

$$\int_{\mathbb{R}^N} G(t, x - \lambda v, v, \xi, \nu) \, dv = \frac{1}{(2\sigma d_\lambda(t))^{N/2}} \mathcal{N}\left(\frac{x - \xi - \eta_\lambda(t)\nu}{\sqrt{2\sigma d_\lambda(t)}}\right).$$

Using these results, let us obtain, in the following lemma, the properties of the operator $G[\mu]$ defined by (2.2). Related estimates in the case of L^p spaces have been deduced by F. Bouchut in [2]. In fact, we prove the same estimate as in [4, Lemma 2] in the case of the modified Morrey spaces.

LEMMA 2.2. *If $\mu \in \tilde{L}_1(\mathbb{R}^{2N}) \cap M\tilde{L}_p(\mathbb{R}^{2N})$, $1 \leq p \leq q \leq \infty$, and C is a constant independent of μ, then the following property holds*

$$\|G[\mu]\|_{M\tilde{L}_q(\mathbb{R}^{2N})} \leq C\, d_0(t)^{-(1/2)(N/p - N/q)} \|\mu\|_{M\tilde{L}_p(\mathbb{R}^{2N})},$$

$$\|\nabla_\nu G[\mu]\|_{M\tilde{L}_q(\mathbb{R}^{2N})} \leq C\, t^{-1/2}\, d_0(t)^{-(1/2)(N/p - N/q)} \|\mu\|_{M\tilde{L}_p(\mathbb{R}^{2N})}.$$

Moreover, for $G[\mu]$ and $p = q$ the constant $C = 1$.

Proof. We prove the estimate for $q = p$ and $q = \infty$; then, using the interpolation property (1.5), we will obtain the result for $p \leq q \leq \infty$.

For $q = p$ we use the definition of the norm in the spaces $M\tilde{L}_p(\mathbb{R}^{2N})$, given by (1.4), and the fundamental solution G (2.1) to have

$$\int_{B(a,r,h)}\int_{\mathbb{R}^{2N}} G_0\big(t, x-\xi-\eta(t)\nu, v-e^{-\beta t}\nu\big)d|\mu|(\xi,\nu)d(x,v)$$
$$= \int_{\mathbb{R}^{2N}}\int_{B(a,r,h)} G_o\big(t, x-\xi-\eta(t)\nu, v-e^{-\beta t}\nu\big)d(x,v)d|\mu|(\xi,\nu), \tag{2.3}$$

where we have applied the Fubini theorem. Making the change of variables $k = x - \xi - \eta(t)\nu$ and $l = v - e^{-\beta t}\nu$, we find

$$\int_{\mathbb{R}^{2N}}\int_{B(a,r,h)} G_0\big(t, x-\xi-\eta(t)\nu, v-e^{-\beta t}\nu\big)d(x,v)d|\mu|(\xi,\nu)$$
$$= \int_{\mathbb{R}^{2N}}\int_{B(a-\xi-\eta(t)\nu-he^{-\beta t}\nu,\, r,\, h)} G_0(t,k,l)d(k,l)d|\mu|(\xi,\nu).$$

Denoting by $\mathscr{X}_A$ the characteristic function of the set A and applying the Fubini theorem we can obtain

$$\int_{\mathbb{R}^{2N}}\int_{\mathbb{R}^{2N}} \mathscr{X}_{B(a-\xi-\eta(t)\nu-he^{-\beta t}\nu,\, r,\, h)}(k,l)d|\mu|(\xi,\nu)G_0(t,k,l)d(k,l)$$
$$= \int_{\mathbb{R}^{2N}} TV(\mu)\big(B\big(a-k-hl, r, \eta(t)+he^{-\beta t}\big)\big)G_0(t,k,l)d(k,l).$$

By the definition of the norm of the space $M\tilde{L}_p(\mathbb{R}^{2N})$, given by (1.4), we deduce

$$\int_{\mathbb{R}^{2N}} TV(\mu)\big(B\big(a-k-hl, r, \eta(t)+he^{-\beta t}\big)\big)G_0(t,k,l)d(k,l)$$
$$\leq \|\mu\|_{M\tilde{L}_p(\mathbb{R}^{2N})} r^{N/p'} \int_{\mathbb{R}^{2N}} G_0(t,k,l)d(k,l).$$

Then, taking into account the property (i) of Lemma 2.1 we deduce the result for $q = p$.

In the case $q = \infty$ we change variables in (2.3) to obtain

$$\int_{B(a,r,h)} \int_{\mathbb{R}^{2N}} G(t, x, v, \xi, \nu)\, d|\mu|(\xi, \nu)\, d(x, v)$$

$$= \int_{\mathbb{R}^{2N}} \int_{B(a,r)} \int_{\mathbb{R}^N} G(t, x - hv, v, \xi, \nu)\, dv\, dx\, d|\mu|(\xi, \nu)$$

$$= \int_{B(a,r)} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^N} G(t, x - hv, v, \xi, \nu)\, dv\, d|\mu|(\xi, \nu)\, dx.$$

Therefore, to obtain a bound in $M\tilde{L}_\infty(\mathbb{R}^{2N})$ is enough to bound in $L^\infty(\mathbb{R}^N)$ independently of h the following function of x,

$$\int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^N} G(t, x - hv, v, \xi, \nu)\, dv\, d|\mu|(\xi, \nu).$$

In order to do this, we use the distribution function of μ and the explicit expression of the integral in v of $G(t, x - hv, v, \xi, \nu)$ given by Lemma 2.1 (iii) to obtain

$$\int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^N} G(t, x - hv, v, \xi, \nu)\, dv\, d|\mu|(\xi, \nu)$$

$$= \frac{1}{\left(2\sigma\, d_h(t)\right)^{N/2}} \int_0^1 TV(\mu)\left(\left\{(\xi, \nu) / e^{-|x - \xi - \eta_h(t)\nu|^2/2\sigma d_h(t)} > s\right\}\right) ds.$$

Thus, we find

$$\int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^N} G(t, x - hv, v, \xi, \nu)\, dv\, d|\mu|(\xi, \nu)$$

$$\leq \frac{1}{\left(2\sigma d_h(t)\right)^{N/2}} \int_0^1 TV(\mu)\Big(\Big\{(\xi, \nu) / |x - \xi - \eta_h(t)\nu|$$

$$< \left(-2\sigma d_h(t) \log s\right)^{1/2}\Big\}\Big)\, ds.$$

Since μ belongs to $M\tilde{L}_p(\mathbb{R}^{2N})$, we have

$$\int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^N} G(t, x - hv, v, \xi, \nu)\, dv\, d|\mu|(\xi, \nu)$$

$$\leq \frac{1}{\left(2\sigma d_h(t)\right)^{N/2}} \|\mu\|_{M\tilde{L}_p(\mathbb{R}^{2N})} \int_0^1 \left(-2\sigma\, d_h(t) \log s\right)^{N/2p'} ds$$

$$= C\left(2\sigma d_h(t)\right)^{-N/2p} \|\mu\|_{M\tilde{L}_p(\mathbb{R}^{2N})} \leq C\, d_0(t)^{-N/2p} \|\mu\|_{M\tilde{L}_p(\mathbb{R}^{2N})},$$

where we have applied that $d_0(t) \le d_h(t)$. Hence, the definition of the space $M\tilde{L}_\infty(\mathbb{R}^{2N})$ implied that

$$\| \mu\|_{M\tilde{L}_\infty(\mathbb{R}^{2N})} \le C\, d_0(t)^{-N/2p} \| \mu\|_{M\tilde{L}_p(\mathbb{R}^{2N})}.$$

Then, using the interpolation property (1.5) the result is proved. The other assertions are deduced directly using the properties of G given in Lemma 2.1(ii). ∎

Let us recall a property of the Morrey spaces in relation to convolution operators. The proof of this property can be found in [7]. We write it in the three-dimensional case.

LEMMA 2.3. *If* $\rho \in \tilde{L}_p(\mathbb{R}^3) \cap \tilde{L}_q(\mathbb{R}^3)$, $1/p + 2/3 < 1 < 1/q + 2/3$, *and C independent of* ρ, *then* $E = K * \rho \in L^\infty(\mathbb{R}^3)^3$ *and*

$$\|E\|_{L^\infty(\mathbb{R}^3)^3} \le C \| \rho\|_{\tilde{L}_p(\mathbb{R}^3)}^{(2/3 - 1/q')/(1/q - 1/p)} \| \rho\|_{\tilde{L}_q(\mathbb{R}^3)}^{(1/p' - 2/3)/(1/q - 1/p)}.$$

3. APPROXIMATING SEQUENCE AND *A PRIORI* ESTIMATES

In order to obtain a weak solution we will use a linearization of Eq. (1.1) by considering the nonlinear term as a second member retarded in time. In this way we will have a sequence of mollified problems. We use this direct linearization instead of the classical regularization due to E. Horst and R. Hunze [8] because in our development it is not necessary to bound uniformly the kinetic energy of the mollified problems to pass to the limit. Moreover, it is not straightforward to pass to the limit with the classical regularization of E. Horst and R. Hunze in Morrey spaces.

Assuming that $f \in C([0, \infty), L^1(\mathbb{R}^6))$ and $E \in L^\infty(0, T; L^\infty(\mathbb{R}^3)^3)$, it can be proved (see [1]) that if f is a weak solution of Eq. (1.1), then f is a solution of the integral equation

$$f(t, x, v) = \int_{\mathbb{R}^6} G(t, x, v, \xi, \nu)\, f_0(\xi, \nu)\, d\xi\, d\nu$$
$$+ \int_0^t \int_{\mathbb{R}^6} \nabla_\nu G(t - s, x, v, \xi, \nu)\, E(s, \xi) f(s, \xi, \nu)\, d\xi\, d\nu\, ds.$$

Moreover, it is a simple matter to verify that any solution of the integral equation with the regularity of Theorem 1.1 is a weak solution of Eq. (1.1). Thus, we are going to consider an interative scheme to approximate the solution of our VPFP problem via the above integral equation.

Define f^0 by $f^0 = G[f_0]$. The iterative scheme is defined, for any $n \geq 0$, as

$$\begin{aligned} f^{n+1}&(t, x, v) \\ &= \int_{\mathbb{R}^6} G(t, x, v, \xi, \nu) f_0(\xi, \nu)\, d\xi\, d\nu \\ &\quad + \int_0^t \int_{\mathbb{R}^6} \nabla_\nu G(t-s, x, v, \xi, \nu) E^n(s, \xi) f^n(s, \xi, \nu)\, d\xi\, d\nu\, ds. \end{aligned} \tag{3.1}$$

The next result proves that this sequence is well-defined and it provides us with some important a priori estimates.

Firstly, let us justify the reason to choose the modified Morrey spaces.

Taking into account the properties of G in Lemma 2.1, we can express f^{n+1} in a different way. From (3.1) we can write f^{n+1} as $f^{n+1} = f_1^{n+1} + f_2^{n+1}$, where f_1^{n+1} and f_2^{n+1} are respectively the first and second terms in (3.1). Let $h \in \mathbb{R}$. Using (1.3) we can express $\rho_h(f_1^{n+1})(t, x)$ as

$$\begin{aligned} \rho_h\left(f_1^{n+1}\right)(t, x) &= \int_{\mathbb{R}^3} f_1^n(t, x - hv, v)\, dv \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^6} G(t, x, v - hv, \xi, \nu) f_0(\xi, \nu)\, d\xi\, d\nu\, dv. \end{aligned}$$

By means of Lemma 2.1(iii), we can write the previous relationship as

$$\rho_h\left(f_1^{n+1}\right)(t, x) = \int_{\mathbb{R}^6} \frac{1}{\left(2\sigma d_h(t)\right)^{3/2}} \mathcal{N}\left(\frac{x - \xi - \eta_h(t)\nu}{\sqrt{2\sigma d_h(t)}}\right) f_0(\xi, \nu)\, d\xi\, d\nu.$$

Finally, making the change of variables $\xi = \xi - \eta_h(t)\nu$, we obtain

$$\begin{aligned} \rho_h&\left(f_1^{n+1}\right)(t, x) \\ &= \int_{\mathbb{R}^3} \frac{1}{\left(2\sigma d_h(t)\right)^{3/2}} \mathcal{N}\left(\frac{x - \xi}{\sqrt{2\sigma d_h(t)}}\right) \\ &\quad \times \int_{\mathbb{R}^3} f_0(\xi - \eta_h(t)\nu, \nu)\, d\nu\, d\xi. \end{aligned}$$

In an analogous way, we can write $\rho_h(f_2^{n+1})(t, x)$ as

$$\begin{aligned} \rho h&\left(f_2^{n+1}\right)(t, x) \\ &= \int_0^t \int_{\mathbb{R}^3} \left\{ \frac{-\eta_h(t-s)}{\left(2\sigma d_h(t-s)\right)^2} \nabla\mathcal{N}\left(\frac{x - \xi}{\sqrt{2\sigma d_h(t-s)}}\right) M_h(s, \xi) \right\} d\xi\, ds, \end{aligned}$$

where

$$M_h(s, \xi) = \int_{\mathbb{R}^3} E^n(s, \xi - \eta_h(t)\nu) f^n(s, \xi - \eta_h(t)\nu, \nu)\, d\nu.$$

Note that in order to estimate $\rho_h(f^{n+1})$ it will be necessary to consider the spaces $M\tilde{L}_p(\mathbb{R}^{2N})$. In fact, even for $\rho_0(f_2^{n+1})$, in our development we must estimate terms of the form $\rho_h(f^n)$ defined in (1.3).

Let us obtain the *a priori* estimates in the spaces $M\tilde{L}_p(\mathbb{R}^6)$ which assure us that the sequence (3.1) is well-defined.

THEOREM 3.1. *If $\pi(f_0)$ denotes the maximum of $\|f_0\|_{M\tilde{L}_{p_0}(\mathbb{R}^6)}$ and $\|f_0\|_{\tilde{L}_1(\mathbb{R}^6)}$ up to a multiplicative constant, with $p_0 \geq 9/4$, then the following estimates are verified for any t which belongs to $(0, T^*)$:*

$$\|f^n(t, \cdot)\|_{M\tilde{L}_{p_0}(\mathbb{R}^6) \cap \tilde{L}_1(\mathbb{R}^6)} \leq C\pi(f_0). \tag{3.2}$$

$$\text{If } p_0 \leq 3, \text{ then } \|E^n(t, \cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leq C\pi(f_0) t^{(3/2 - 9/2p_0)}. \tag{3.3}$$

$$\text{If } p_0 \geq 3, \text{ then } \|E^n(t, \cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leq C\pi(f_0). \tag{3.4}$$

T^* depends on $\pi(f_0)$ and on the constants σ and β. $T^* = \infty$ for small initial data when $p_0 = 9/4$. C is a constant independent of n and $\pi(f_0)$.

Proof. We will use Lemmas 2.2 and 2.3 to estimate the $L^\infty(\mathbb{R}^3)^3$ norm of the field E. We consider first the case $p_0 \leq 3$. With this aim we choose a number q between $3/2$ and 3 such that $q < p_0 < 2q$. Then, using (3.1), we will derive certain estimates for $\|f^n(t, \cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)}$, with $p = q, 2q, p_0$, and 1. The last part of the proof is devoted to show that the norms $\|f^n(t, \cdot)\|_{M\tilde{L}_{p_0}(\mathbb{R}^6)}$ and $\|E^n(t, \cdot)\|_{L^\infty(\mathbb{R}^3)^3}$ remain bounded in a certain time interval $(0, T^*)$.

Using Lemma 2.3 and the fact that the $\tilde{L}_p(\mathbb{R}^3)$ norm of $\rho(f^n)(t, \cdot)$ is bounded by the $M\tilde{L}_p(\mathbb{R}^6)$ norm of $f^n(t, \cdot)$, we obtain, for $\theta = 2q/3$, the estimate

$$\|E^n(t, \cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leq C \|f^n(t, \cdot)\|^{\theta - 1}_{M\tilde{L}_q(\mathbb{R}^6)} \|f^n(t, \cdot)\|^{2-\theta}_{M\tilde{L}_{2q}(\mathbb{R}^6)}. \tag{3.5}$$

Lemma 2.2 and the expression of f_1^{n+1}, given by (3.1), allow us to find

$$\|f_1^{n+1}(t, \cdot)\|_{M\tilde{L}_{2q}(\mathbb{R}^6)} \leq C d_0(t)^{(3/4q - 3/2p_0)} \|f_0\|_{M\tilde{L}_{p_o}(\mathbb{R}^6)}.$$

To estimate f_2^{n+1}, which is defined by (3.1), we use Lemma 2.2 again with the operator $\nabla_v G$ to obtain

$$\|f_2^{n+1}(t, \cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)} \leq C \int_0^t (t-s)^{-1/2} \|E^n f^n(s, \cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)}\, ds.$$

From this inequality, it is a simple matter to deduce that

$$\|f_2^{n+1}(t,\cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)} \le C\int_0^t (t-s)^{-1/2} \max_{0\le\tau\le T}\left\{\|f^n(\tau,\cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)}\|E^n(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)^3}\right\} ds. \tag{3.6}$$

Inequality (3.6) is verified for all $1 \le p \le \infty$. Combining (3.6) and Lemma 2.2 we can obtain the following estimates for $p = p_0$, q, and 1,

$$\|f^{n+1}(t,\cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)} \le \|f_0\|_{M\tilde{L}_p(\mathbb{R}^6)} + C\int_0^t (t-s)^{-1/2} \times \max_{0\le\tau\le T}\left\{\|f^n(\tau,\cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)}\|E^n(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)^3}\right\} ds,$$

$$\|f^{n+1}(t,\cdot)\|_{M\tilde{L}_{2q}(\mathbb{R}^6)} \le C\,d_0(t)^{(3/4q-3/2p_0)}\|f_0\|_{M\tilde{L}_{p_0}(\mathbb{R}^6)} + C\int_0^t (t-s)^{-1/2} \times \max_{0\le\tau\le T}\left\{\|f^n(\tau,\cdot)\|_{M\tilde{L}_{2q}(\mathbb{R}^6)}\|E^n(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)^3}\right\} ds.$$

Note that, due to the interpolation property (1.5), $\|f_0\|_{M\tilde{L}_q(\mathbb{R}^6)}$ is bounded for $1 < q < p_0$.

Set $K_p^0 = \|f_0\|_{M\tilde{L}_p(\mathbb{R}^6)}$, $K_p^n = \max\{K_p^n(t),\, 0 \le t \le T\}$ with

$$K_p^n(t) = \|f^n(t,\cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)}\, d_0(t)^{(3/2p_0-3/2p)}, \qquad \text{for } p > p_0$$

and

$$K_p^n(t) = \|f^n(t,\cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)}, \qquad \text{for } p \le p_0.$$

We can rewrite the above estimates in the following way

$$K_{p_0}^{n+1}(t) \le K_{p_0}^0 + C\int_0^t (t-s)^{-1/2} d_0(s)^{(3/4q-3/2p_0)(2-\theta)} \times ds\, K_{p_0}^n\left[K_q^n\right]^{\theta-1}\left[K_{2q}^n\right]^{2-\theta},$$

$$K_1^{n+1}(t) \le K_1^0 + C\int_0^t (t-s)^{-1/2}\, d_0(s)^{(3/4q-3/2p_0)(2-\theta)} \times ds\, K_1^n\left[K_q^n\right]^{\theta-1}\left[K_{2q}^n\right]^{2-\theta},$$

$$K_q^{n+1}(t) \le K_q^0 + C\int_0^t (t-s)^{-1/2}\, d_0(s)^{(3/4q-3/2p_0)(2-\theta)} \times ds\left[K_q^n\right]^{\theta}\left[K_{2q}^n\right]^{2-\theta},$$

and

$$K_{2q}^{n+1}(t) \leq K_{p_o}^{o} + C\, d_0(t)^{(3/2p_0 - 3/4q)} \int_0^t (t-s)^{-1/2}\, d_0(s)^{(3/4q - 3/2p_0)(3-\theta)}$$
$$\times\, ds \left[K_q^n\right]^{\theta-1} \left[K_{2q}^n\right]^{3-\theta}.$$

Let $B(a, b)$ be the beta function defined by

$$B(a,b) = \int_0^t (t-s)^{a-1} s^{b-1}\, ds.$$

It is known that if $a, b > 0$, then $B(a,b) = C(a,b)\, t^{a+b-1}$, where $C(a,b)$ is a constant which depends on a and b. Using that $d_0(t)$ has a behavior in 0 of order t^3, it is easy to deduce, for any $a > 0$ and $3c + 1 > 0$, that

$$\int_0^t (t-s)^{a-1} d_0(s)^c\, ds \leq CB(a, 3c+1).$$

Thus, if we assume that

$$1 + \left(\frac{9}{4q} - \frac{9}{2p_0}\right)(2-\theta) > 0 \qquad \text{and} \qquad 1 + \left(\frac{9}{4q} - \frac{9}{2p_0}\right)(3-\theta) > 0,$$

the above beta functions converge. It can be easily verified that for $p_0 > 9/4$ and chosen q close enough to $3/2$ these estimates are verified. If we set

$$B = \max_{0 \leq t \leq T} \left\{ B\left(\frac{1}{2}, 1 + \left(\frac{9}{4q} - \frac{9}{2p_0}\right)(2-\theta)\right), \right.$$
$$\left. d_0(t)^{(3/2p_0 - 3/4q)} B\left(\frac{1}{2}, 1 + \left(\frac{9}{4q} - \frac{9}{2p_0}\right)(3-\theta)\right) \right\}$$

and $K^n = \max\{K_{p_0}^n, K_1^n, K_q^n, K_{2q}^n\}$, then $K^{n+1} \leq \pi(f_0) + CB[K^n]^2$.

Now, let us denote by K the supremum of all the K^n for every n. Thus,

$$k \leq \pi(f_0) + CBK^2. \tag{3.7}$$

Therefore, taking $\pi(f_0) \leq 1/4CB$ we find

$$K \leq \frac{1 - \left(1 - 4CB\pi(f_0)\right)^{1/2}}{2CB} = \lambda < \frac{1}{2CB}. \tag{3.8}$$

Then, estimate (3.2) is proved. Using (3.5), (3.2), and the definition of K_{2q}^n we an obtain

$$\|E^n(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leq d_0(t)^{(3/4q-3/2p_0)(2-\theta)} 2\pi(f_0).$$

Taking into account the behavior of $d_0(t)$ at zero and getting $q \to 3/2$, the estimate on E is proved. This completes the proof in the case $p_0 \leq 3$.

If $p_0 > 3$, the proof can be derived from (3.5), (3.6), and from the interpolation property (1.5). In this way, we obtain the estimate $K \leq \pi(f_0) + CT^{1/2} K^2$. Then, for $\pi(f_0) \leq 1/4CT^{1/2}$ the proof is completed analogously.

Let us make a similar study in the case $p_0 = 9/4$. Since the steps of the proof are basically the same as above, we only sketch the proof. In order to simplify the proof we can write t^3 instead of $d_0(t)$ without loss of generality. Set q between $9/4$ and 3, then $4q/3$ takes its values between 3 and 4. Using Lemma 2.3 we obtain, for $\alpha = 4q/3 - 3$, that

$$\|E^n(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leq C\|f^n(t,\cdot)\|^{\alpha}_{M\tilde{L}_q(\mathbb{R}^6)} \|f^n(t,\cdot)\|^{1-\alpha}_{M\tilde{L}_{4q/3}(\mathbb{R}^6)}. \tag{3.9}$$

Using Lemma 2.2 we obtain

$$\begin{aligned}\|f^{n+1}(t,\cdot)\|_{M\tilde{L}_{4q/3}(\mathbb{R}^6)} &\leq Ct^{(27/8q-2)}\|f_0\|_{M\tilde{L}_{9/4}(\mathbb{R}^6)} \\ &\quad + C\int_0^t (t-s)^{-1/2-(9/2)(1/r-3/4q)} \max_{0\leq\tau\leq T} \\ &\quad \times \left\{\|f^n(\tau,\cdot)\|_{M\tilde{L}_r(\mathbb{R}^6)}\|E^n(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)^3}\right\} ds,\end{aligned}$$

where $r \leq 4q/3$. The interpolation property (1.5) allows us to deduce

$$\|f^n(\tau,\cdot)\|_{M\tilde{L}_r(\mathbb{R}^6)} \leq \|f^n(\tau,\cdot)\|^{\alpha'}_{M\tilde{L}_{9/4}(\mathbb{R}^6)} \|f^n(\tau,\cdot)\|^{1-\alpha'}_{M\tilde{L}_{4q/3}(\mathbb{R}^6)},$$

where α' is given by

$$\frac{4\alpha'}{9} + \frac{3(1-\alpha')}{4q} = \frac{1}{r}.$$

Using the above estimates, the notation stated above, and taking into account the equalities

$$\left(\frac{27}{8q} - 2\right)(1-\alpha') = \frac{9}{2r} - 2$$

$$\text{and} \qquad 1 + \left(\frac{9}{2q} - 2\right)\alpha + \left(\frac{27}{8q} - 2\right)(1-\alpha) = \frac{1}{2},$$

we can rewrite the above estimates in the following way

$$K_{9/4}^{n+1}(t) \le \pi(f_0) + CB\left(\frac{1}{2}, \frac{1}{2}\right) K_{9/4}^n \left[K_q^n\right]^{\alpha} \left[K_{4q/3}^n\right]^{1-\alpha},$$

$$K_q^{n+1}(t) \le \pi(f_0) + Ct^{(2-9/2q)} B\left(\frac{1}{2}, -\frac{3}{2} + \frac{9}{2q}\right) \left[K_q^n\right]^{1+\alpha} \left[K_{4q/3}^n\right]^{1-\alpha},$$

and

$$K_{4q/3}^{n+1}(t) \le \pi(f_0) + Ct^{(2-27/8q)} B\left(\frac{1}{2} + \frac{27}{8q} - \frac{9}{2r}, -\frac{3}{2} + \frac{9}{2r}\right) \times \left[K_{9/4}^n\right]^{\alpha'} \left[K_q^n\right]^{\alpha} \left[K_{4q/3}^n\right]^{2-\alpha-\alpha'}.$$

Thus, if we assume that

$$-\frac{3}{2} + \frac{9}{2q} > 0, \qquad \frac{1}{2} + \frac{27}{8q} - \frac{9}{2r} > 0, \qquad \text{and} \qquad -\frac{3}{2} + \frac{9}{2r} > 0,$$

the above beta functions converge. For q fixed, it can be easily verified that, for $36/13 < r < 3$, the above estimates hold. Set

$$B(T) = \max\left\{ B\left(\frac{1}{2}, \frac{1}{2}\right), t^{(2-9/2q)} B\left(\frac{1}{2}, -\frac{3}{2} + \frac{9}{2q}\right), t^{(2-27/8q)} \times B\left(\frac{1}{2} + \frac{27}{8q} - \frac{9}{2r}, -\frac{3}{2} + \frac{9}{2r}\right), 0 \le t \le T \right\}$$

and $K = \max\{K_{9/4}^n, K_q^n, K_{4q/3}^n\}$. Now, using the definition of the beta function, it is straightforward to deduce that $B(T)$ does not depend on T, i.e., $B(T)$ is a constant $B(T) = B$ depending on q and r. Therefore, we find $K \le \pi(f_0) + CBK^2$ which is the same equation as (3.7). Thus, for small initial data, i.e., $\pi(f_0) \le 1/4CB$, we obtain a global in time estimate with $p_0 = 9/4$. Since $1 + (9/2q - 2)\alpha + (27/8q - 2)(1 - \alpha) = 1/2$, we finally deduce, for any $t > 0$, $\|E(t, \cdot)\|_{L^\infty(\mathbb{R}^3)^3} \le Ct^{-1/2}\pi(f_0)$. ∎

The last result implies that the sequence defined by (3.1) is well-defined under the regularity of the initial data which have been assumed in our main result (Theorem 1.1). Moreover, we can obtain better properties of the sequence of modified densities f^n and force fields E^n using the smoothing effect of the operator L (see [2]) and the existence properties of the linear equation (see [13]). However, the bounds are not independent of n. The next proposition is an easy consequence of these results. Let us denote by $S_p(f^n)(t)$ the quantity

$$S_p(f^n)(t) = \max\{\|\rho_h(f^n)(t,\cdot)\|_{L^p(\mathbb{R}^3)}, h \ge 0\},$$

for any $1 \le p \le \infty$ and $t > 0$.

PROPOSITION 3.2. *Assume that f_0 verifies the hypotheses of Theorem* 1.1. *The functions f^n and E^n have the following regularity properties*:

(i) *f^n is C^1 respect to (t, x) and C^2 with respect to v on $(0, \infty) \times \mathbb{R}^6$. Moreover, f^n and their derivatives are bounded on $[\epsilon, T] \times \mathbb{R}^6$ with $0 < \epsilon < T < \infty$, they verify $Lf^{n+1} = -(E^n\nabla_v)f^n$, and $f^n \in C((0, \infty), L^1 \cap L^\infty(\mathbb{R}^6))$.*

(ii) $S_p(f^n)(t)$ is bounded on $[\epsilon, T]$ with $0 < \epsilon < T < \infty$, for any $t > 0$, $1 \le p \le \infty$.

(iii) $E^n \in C([\epsilon, T], W^{1,\infty}(\mathbb{R}^6))$ with $0 < \epsilon < T < \infty$.

(iv) The total mass of the system is preserved, for any $t > 0$, that is, $\|f^n(t,\cdot)\|_{L^1(\mathbb{R}^6)} = \|f_0\|_{\mathscr{M}(\mathbb{R}^6)}$.

The next section is devoted to the proof of Theorem 1.1

4. EXISTENCE, UNIQUENESS, AND STABILITY OF SOLUTIONS

Firstly, let us prove that $\{f^n\}$, $\{E^n\}$ are Cauchy sequences in suitable spaces.

LEMMA 4.1. *The following statements are verified*:

(i) *$\{f^n\}$ is a Cauchy sequence in the space $L^\infty(0, T; \tilde{L}_1(\mathbb{R}^6) \cap M\tilde{L}_{p_0}(\mathbb{R}^6))$.*

(ii) *The sequence $\{t^{(9/2p_0 - 3/2)}E^n\}$ is a Cauchy sequence in $L^\infty(0, T; L^\infty(\mathbb{R}^3)^3)$, for $p_0 < 3$.*

(iii) *$\{E^n\}$ is a Cauchy sequence in $L^\infty(0, T; L^\infty(\mathbb{R}^3)^3)$, for $p_0 \ge 3$.*

(iv) *$\{f^n\}$ is a Cauchy sequence in $BC((0, T], L^1(\mathbb{R}^6) \cap ML_{p_0}(\mathbb{R}^6))$ and also in the space $BC((0, T], ML_r(\mathbb{R}^6))$, with $r > 3$ close enough to* 3.

(v) *$\{E^n\}$ is a Cauchy sequence in the space $BC((0, T], C(\mathbb{R}^3)^3)$.*

Proof. Let us use the notation and techniques introduced in Theorem 3.1. We consider the difference $f^{n+1} - f^n$ in the norm of the spaces $M\tilde{L}_p(\mathbb{R}^6)$ for $p = p_0, q, 2q$, and 1. We can estimate these norms as

$$\|(f^{n+1} - f^n)(t,\cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)}$$

$$\le C\int_0^t (t-s)^{-1/2}\|(E^nf^n)(s,\cdot) - (E^{n-1}f^{n-1})(s,\cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)}\, ds.$$

Now, introducing the term $(E^n f^{n-1})(s, \cdot)$ in the above expression we deduce

$$\begin{aligned}
&\|(f^{n+1} - f^n)(t, \cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)} \\
&\quad \le C\int_0^t (t-s)^{-1/2} \max_{0\le\tau\le T} \\
&\qquad \times \left\{\|(f^n - f^{n-1})(\tau, \cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)} \|E^n(\tau, \cdot)\|_{L^\infty(\mathbb{R}^3)^3}\right\} ds \\
&\qquad + C\int_0^t (\sigma(t-s))^{-1/2} \max_{0\le\tau\le T} \\
&\qquad \times \left\{\|f^{n-1}(\tau, \cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)} \|(E^n - E^{n-1})(\tau, \cdot)\|_{L^\infty(\mathbb{R}^3)^3}\right\} ds.
\end{aligned}$$

Following a similar reasoning as in Theorem 3.1, we set

$$\begin{aligned}
&J_p^n(t) = \|(f^{n+1} - f^n)(t, \cdot)\|_{M\tilde{L}_p(\mathbb{R}^6)}, \qquad J_p^n = \max\left\{J_p^n(t), 0 \le t \le T\right\}, \\
&J^n(t) = \max\left\{J_{p_0}^n(t), J_1^n(t), J_q^n(t), J_{2q}^n(t)\right\} \qquad \text{and} \\
&\quad J^n = \max\{J^n(t), 0 \le t \le T\}.
\end{aligned}$$

Then, applying the same steps as in Theorem 3.1, we have

$$J^n \le 2CBKJ^{n-1}. \tag{4.1}$$

Iterating the previous bound we obtain $J^n \le (2CBK)^{n-1}J^1$. Therefore, using (3.8) the lemma is proved. ∎

Now, we can finish the proof of the main result, Theorem 1.1.

Proof of Theorem 1.1. Let us prove that the limits of these sequences, denoted by E and f, respectively, are a weak solution of the VPFP problem and a solution of the integral equation (3.1).

Since $\{f^n\}$ converges to f in $L^\infty(0, T; \tilde{L}_1(\mathbb{R}^6) \cap M\tilde{L}_{p_0}(\mathbb{R}^6))$ we deduce that $\{\rho(f^n)\}$ converges to $\rho(f)$ in $L^\infty(0, T; \tilde{L}_1(\mathbb{R}^3) \cap \tilde{L}_{p_0}(\mathbb{R}^3))$. Using Lemma 2.2 and the arguments of Theorem 3.1 we can obtain that the sequence $\{K * \rho(f^n)\} = \{E^n\}$ converges to $K * \rho(f)$ in $L^\infty(\mathbb{R}^3)^3$, for every $t > 0$. Thus, $E = K * \rho(f)$.

Since f^{n+1} is a solution of the integral equation (3.1), similar arguments as those done in Lemma 4.1 imply that the integral formula is verified by the solution f, i.e.,

$$\begin{aligned}
f(t, x, v) &= \int_{\mathbb{R}^6} G(t, x, v, \xi, \nu) f_0(\xi, \nu)\, d\xi\, d\nu \\
&\quad + \int_0^t \int_{\mathbb{R}^6} \nabla_v G(t-s, x, v, \xi, \nu) E(s, \xi) f(s, \xi, \nu)\, d\xi\, d\nu\, ds.
\end{aligned}$$

As a consequence, f is a weak solution of the VPFP problem.

Finally, we must prove that $f(t)$ goes to f_0 when t goes to 0 weak-★ in the space of bounded measures. Using (3.1) and the fact that G is the fundamental solution of the linear part of (1.1) we obtain that $f(t)$ converges to f_0 in the distribution topology. This fact and the boundedness of $f(t)$ in $\tilde{L}_1(\mathbb{R}^6)$ ensure us that $f(t)$ goes to f_0, when t goes to 0, in the weak-★ topology of bounded measures.

To finish the proof of Theorem 1.1, we have to show that f is positive. This is a consequence of the properties of corresponding linear problem (see [1]) and a density argument in the weak-★ topology of bounded measures. ∎

Remark 1. Reiterating k times the same estimates than in Theorem 3.1, it can be proved easily that the solution which has been obtained exists, with our arguments, at least in the interval $(0, T^*)$ with

$$T^* = \frac{1}{\left(4C\pi(f_0)\right)^{2p_0/(4p_0-9)}} \sum_{n=1}^{\infty} \left(2^{2p_0/(4p_0-9)}\right)^{-n}.$$

Note that due to the regularity deduced in Theorem 1.1 and a similar reasoning, applied to the difference between two solutions of the VPFP system, as in Theorem 3.1 and Lemma 4.1, we can deduce the uniqueness of solution for this problem. Moreover, we can prove the following result of stability which includes the uniqueness as a particular case.

COROLLARY 4.2. *Assume that* $f_0,\ \tilde{f}_0 \in \tilde{L}_1(\mathbb{R}^6) \cap M\tilde{L}_{p_0}(\mathbb{R}^6)$, *with* $p_0 \geq 9/4$. *If* f *and* $\tilde{f}$ *are solutions of the integral equation* (3.1) *with initial data* f_0 *and* $\tilde{f}_0$ *which verify the regularity of Theorem* 1.1, *then there exists a time interval* $(0, T^*)$ *such that for any* $t \in (0, T^*)$ *we have*

$$\|f(t,\cdot) - \tilde{f}(t,\cdot)\|_{\tilde{L}_1(\mathbb{R}^6)} \leq C\pi\left(f_0 - \tilde{f}_0\right),$$

$$\|f(t,\cdot) - \tilde{f}(t,\cdot)\|_{M\tilde{L}_{p_0}(\mathbb{R}^6)} \leq C\pi\left(f_0 - \tilde{f}_0\right).$$

$$\text{If } p_0 \leq 3, \text{ then } \quad \|E(t,\cdot) - \tilde{E}(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leq C\pi\left(f_0 - \tilde{f}_0\right) t^{-(9/2p_0 - 3/2)}.$$

$$\text{If } p_0 \geq 3, \text{ then } \quad \|E(t,\cdot) - \tilde{E}(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leq C\pi\left(f_0 - \tilde{f}_0\right).$$

Here, $\pi(f_0 - \tilde{f}_0)$ *is the maximum of the norms of* $f_0 - \tilde{f}_0$ *in the spaces* $\tilde{L}_1(\mathbb{R}^6)$ *and* $M\tilde{L}_{p_0}(\mathbb{R}^6)$. $T^* = \infty$ *for small initial data.*

We want to point out that we have obtained from Theorem 3.1 an estimate for the time decay of the $L^\infty(R^3)^3$ norm of E.

COROLLARY 4.3. *Assume that* (E, f) *is the weak solution of the VPFP problem with initial data* $f_0 \in \tilde{L}_1(\mathbb{R}^6) \cap M\tilde{L}_{p_0}(\mathbb{R}^6)$, *with* $p_0 \geq 9/4$, *then for* t

large enough we have

$$\|E(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \le C\pi(f_0)t^{-1/2}.$$

In the next section we are going to connect our results with the classical hypotheses of boundedness of moments.

5. REMARKS ON GLOBAL EXISTENCE

One of the classical ideas in order to obtain global existence for this type of equations has been to assume that some moments are bounded. Let us study the relation between these kind of hypotheses and the hypothesis used here, the Morrey-type hypothesis.

Firstly, we want to point out some results connecting the modified Morrey spaces and the L^p spaces. The following interpolation result is classical (see [8]).

LEMMA 5.1. *If* $f \in L^p(\mathbb{R}^3)$, $1 < p \le \infty$, *with* $f \ge 0$, *and* $J_k = |v|^k f(v) \in L^1(\mathbb{R}^3)$, *then*

$$\|f\|_{L^1(\mathbb{R}^3)} \le C\|f\|_{L^p(\mathbb{R}^3)}^{kp'/(3+kp')}\|J\|_{L^1(\mathbb{R}^3)}^{3/(3+kp')}.$$

The next result is a straightforward consequence of the previous one.

LEMMA 5.2. *If* $f_0 \in L^1(\mathbb{R}^6) \cap L^p(\mathbb{R}^6)$, $p > 9/4 = q_0$, *with* $f_0 \ge 0$, *and* $J_k = |v|^k f_0(x,v) \in L^1(\mathbb{R}^6)$, *for some* $k > 3((p-1)(q_0-1)/(p-q_0)$, *then* $\rho_h(f_0) \in L^r(\mathbb{R}^3)$ *with*

$$r = \frac{3(p-1)+kp}{3(p-1)+k}, \text{ for } p \ne \infty; \qquad r = \frac{3+k}{3}, \text{ for } p = \infty,$$

and

$$\|\rho_h(f_0)\|_{L^r(\mathbb{R}^3)} \le C\|f_0\|_{L^p(\mathbb{R}^3)}^{kp'/(3+kp')}\|J_k\|_{L^1(\mathbb{R}^3)}^{3/(3+kp')},$$

with C *independent of* h. *Consequently,* $f_0 \in M\tilde{L}_r(\mathbb{R}^6)$.

As a consequence, we obtain a local existence and uniqueness result with initial data in L^p spaces.

COROLLARY 5.3. *The VPFP problem with an initial data* $f_0 \in L^1(\mathbb{R}^6) \cap L^p(\mathbb{R}^6)$, *with* $p > 9/4 = q_0$, *and* $J_k = |v|^k f_0(x, v) \in L^1(\mathbb{R})^6)$, *for some* $k > 3((p-1)(q_0-1)/(p-q_0))$, *has a unique weak solution* (E, f) *in a time interval* $[0, T]$, *with* $t > T^*$, *which verifies the properties given in Theorem* 1.1. *Moreover, if the norm of* f_0 *is small enough in* $L^p(\mathbb{R}^6)$, *then the weak solution is global with the properties given in Theorem* 1.1.

Finally, let us point out that Lemma 5.2, Theorem 1 in [1], and Theorem 1.1 can be easily used to derive the following result about global existence.

COROLLARY 5.4. *If f_0 satisfies $f_0 \in L^1(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$ and the moments in v are bounded until $k < m$, with $m > 15/4$, $|v|^k f_0(x, v) \in L^1(\mathbb{R}^6)$, then there exists a unique solution (E, f) of the VPFP problem such that*

$$f \in C\big([0, T], L^1(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)\big) \qquad \text{and}$$

$$t^{1/2}E \in BC\Big((0,T], L^\infty(\mathbb{R}^3)^3\Big).$$

Also, E belongs to $C((0, T], L^\infty(\mathbb{R}^3)^3)$, for any $T > 0$.

The previous corollary complements the result in [1] in which the boundedness of E for positive time is only proved when we assume that the moments are bounded until order $k < m$, $m > 6$.

ACKNOWLEDGMENTS

This work was partially sponsored by DGICYT-MEC (Spain), Proyecto PB92-0953, and DGES (Spain), Proyecto PB95-1203, and E.U. Human Capital and Mobility Programme Contract ERBCHRXCT930413.

REFERENCES

1. F. Bouchut, Existence and uniqueness of a global smooth solution for the VPFP system in three dimensions, *J. Funct. Anal.* **111**, (1993), 239–258.
2. F. Bouchut, Smoothing effect for the non-linear VPFP system, *J. Differential Equations*, **122**, (1995), 225–238.
3. J. A. Carillo and J. Soler, On the initial value problem for the VPFP system with initial data in L^p spaces, *Math. Methods Appl. Sci.* **18**, (1995), 825–839.
4. J. A. Carrillo, J. Soler, and J. L. Vázquez, Asymptotic behaviour for the three dimensional Vlasov-Poisson-Fokker-Planck system, *C. R. Acad. Sci. Paris* **321** (1995), 1195–1200.
5. J. A. Carrillo, J. Soler, J. L. Vázquez, Asymptotic behaviour and selfsimilarity for the three dimensional Vlasov-Poisson-Fokker-Planck system, *J. Funct. Anal.*, **141**, (1996), 99–132.
6. G. H. Cottet and J. Soler, Three-dimensional Navier–Stokes equations for singular filament data, *J. Differential Equations* **74** (1988), 234–253.
7. Y. Giga and T. Miyakawa, Navier–Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces, *Comm. Partial Differential Equations* **14** (1989), 577–618.
8. E. Horst and R. Hunze, Weak solutions of the initial value problem for the unmodified nonlinear Vlasov equation, *Math. Methods Appl. Sci.* **6** (1984), 262–279.
9. A. Majda, G. Majda, and Y. Zheng, Concentrations in the one-dimensional Vlasov-Poisson equations. I. Temporal development and non-unique weak solutions in the single component case, *Phys. D* **74** (1994), 268–300.
10. G. Rein and J. Weckler, Generic global classical solutions of the Vlasov–Fokker–Planck–Poisson system in three dimensions, *J. Differential Equations* **99** (1992), 59–77.

11. J. Soler, L^∞ stability for weak solutions of the Navier–Stokes equations in $\mathbb{R}^3$ with singular initial data in Morrey spaces, *J. Math. Anal. Appl.* **187** (1994), 513–525.
12. H. D. Victory, On the existence of global weak solutions for VPFP Systems, *J. Math. Anal. Appl.* **160** (1991), 515–553.
13. H. D. Victory and B. P. O'Dwyer, On classical solutions of VPFP systems, *Indiana Univ. Math. J.* **39** (1990), 105–157.
14. Y. Zheng and A. Majda, Existence of global weak solutions to one-component Vlasov–Poisson and Fokker–Planck–Poisson systems in one space dimension with measures as initial data, *Comm. Pure Appl. Math.* **47** (1994), 1365–1401.